Int. J. Solids Structures, 1971, Vol. 7, pp. 1627 to 1640. Pergamon Press. Printed in Great Britain

TRANSMISSION OF CONCENTRATED FORCES INTO PRISMATIC SHELLS—I

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Abstract—The work deals with the membrane and bending fields induced in prismatic shells by concentrated forces that are applied at the joints or ridges between two plates. Since many of the basic questions have little to do with supports or finite boundaries, an unbounded prismatic shell consisting of two semi-infinite plates is considered. This geometry lends itself to a straightforward analysis by Fourier transforms, and the results can be expressed in terms of exponential integrals of complex arguments for which tables are available. When the concentrated force lies in the plane of symmetry and is perpendicular to the ridge, an unexpected result is that the membrane forces remain bounded at the load point. In contrast, the moments and transverse shearing forces are singular at the point of loading. The asymptotic forms of the far fields show that the membrane forces decay more slowly than the moments and shearing forces. Thus the orders of the near and far fields reveal that the symmetrically applied perpendicular force is transferred into the prismatic shell principally through bending, and diffused far away through membrane action.

INTRODUCTION

TRANSMISSION of loads in prismatic shells involves the interaction between membrane and bending fields. If the conditions of equilibrium are written in the undeformed configuration, as is normally done to make the analysis mathematically feasible in problems not involving questions of stability, there is no coupling between extension and bending in the differential equations. The two fields are coupled, however, in the boundary conditions that must be satisfied at the joints between the individual plates.

Prismatic shells because of their widespread applications have received considerable attention in the technical literature, as attested by a survey conducted under the auspices of the ASCE [1] in 1963. Nevertheless, relatively few specific cases have been treated by satisfying the differential equations and boundary conditions exactly [2–5], and the majority of the investigations appear to deal with approximate techniques for design purposes.

One of the most fundamental problems in any situation governed by linear differential equations is the study of fields induced by certain point "sources". Such point sources in the theory of plates and shells are concentrated loads. Surprisingly, no singular solutions involving concentrated forces or other actions at a point have been derived and investigated for prismatic shells. Although in some of the previous work [3–5] column reactions have been incorporated as line loads, and concentrated forces could perhaps in principle be obtained as limiting cases, these solutions are in the form of series that involve boundary conditions at transverse supports, and they do not readily yield answers to many pertinent questions.

The present work deals with the membrane and bending fields induced in prismatic shells by concentrated forces that are applied at a joint or ridge between two plates. Since many of the basic questions have little to do with supports or finite boundaries, it suffices at first to consider an unbounded prismatic shell consisting of two semi-infinite plates, as shown in Fig. 1(a). This geometry lends itself to a straightforward analysis by Fourier transforms, while the concentrated forces can be incorporated in the formulation through delta functions. It should be realized at the outset, however, that the present analysis is probably incapable of giving reliable numerical results very near the ridge. Not even speaking about the obvious oversimplifications used in this work if the problem is viewed from the point of three-dimensional elastostatics, the coupling of the extentional and flexural fields at the ridge is also problematic within the theories of plane stress and bending of thin plates. This is particularly so because two of the boundary conditions involve the supplemented or Kirchhoff shearing forces. It appears reasonable to expect, on the other hand, that the analysis will predict the correct trends even in the vicinity of the ridge.

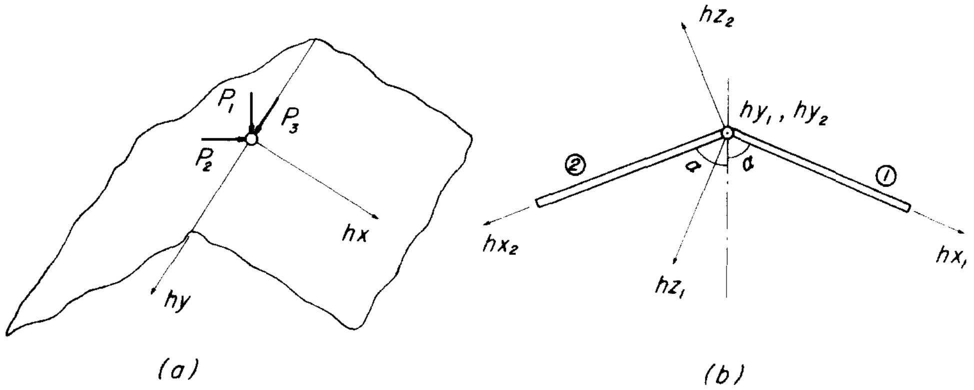

FIG. 1. Unbounded prismatic shell, loads and coordinate systems.

Any of the common representations of the membrane field, such as the Airy stress function or the Papkovich–Neuber displacement potentials are suitable for the problem considered. As the boundary conditions at the ridge involve extensional displacements, there may be a slight advantage, however, in using the displacement potentials. Placing the coordinate system in relation to the prismatic shell as shown in Fig. 1, and measuring distances in terms of the thickness h of the individual plates, the extensional displacements are computed from the displacement potentials $\varphi(x, y)$ and $\psi(x, y)$ as

$$2Gu_x = \frac{3-\nu}{1+\nu}\varphi - x\frac{\partial\varphi}{\partial x} - \frac{1}{h}\frac{\partial\psi}{\partial x},$$
$$2Gu_y = -x\frac{\partial\varphi}{\partial y} - \frac{1}{h}\frac{\partial\psi}{\partial y}, \tag{1}$$

where G and ν denote the shear modulus and Poisson's ratio, respectively. Equations (1) are simply the general expressions connecting displacements and displacement potentials [6] that have been specialized for plane stress. It may be noted that x and y are dimensionless, and that distances along the axes are hx and hy. The membrane forces follow from Hooke's law as

$$N_{xx} = \frac{2}{1+\nu}\frac{\partial\varphi}{\partial x} - x\frac{\partial^2\varphi}{\partial x^2} - \frac{1}{h}\frac{\partial^2\psi}{\partial x^2},$$
$$N_{xy} = \frac{1-\nu}{1+\nu}\frac{\partial\varphi}{\partial y} - x\frac{\partial^2\varphi}{\partial x\partial y} - \frac{1}{h}\frac{\partial^2\psi}{\partial x\partial y}, \tag{2}$$
$$N_{yy} = \frac{2\nu}{1+\nu}\frac{\partial\varphi}{\partial x} - x\frac{\partial^2\varphi}{\partial y^2} - \frac{1}{h}\frac{\partial^2\psi}{\partial y^2}.$$

For equilibrium of the membrane forces, the displacement potentials must be plane harmonic functions, or

$$\nabla^2\varphi(x, y) = \nabla^2\psi(x, y) = 0. \tag{3}$$

The bending field can be specified through the transverse deflection $w(x, y)$ of the middle plane of the plate. The moments and the transverse shearing forces are derived from $w(x, y)$ using the formulas [7]

$$M_{xx} = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right), \text{ etc.,}$$
$$Q_x = -D\frac{\partial}{\partial x}\nabla^2 w, \text{ etc.,} \tag{4}$$

where $D = Gh^3/6(1-\nu)$ is the flexural rigidity. Since no distributed transverse loads are considered, $w(x, y)$ is a plane biharmonic function, or

$$\nabla^4 w(x, y) = 0. \tag{5}$$

For the directions of the coordinate axes shown in Fig. 1(b), the boundary conditions at the ridge are

$$(u_x^{(1)} + u_x^{(2)})\sin\alpha - (w^{(1)} - w^{(2)})\cos\alpha = 0, \tag{6}$$
$$(u_x^{(1)} - u_x^{(2)})\cos\alpha + (w^{(1)} + w^{(2)})\sin\alpha = 0, \tag{7}$$
$$u_y^{(1)} - u_y^{(2)} = 0, \tag{8}$$
$$\frac{\partial w^{(1)}}{\partial x_1} - \frac{\partial w^{(2)}}{\partial x_2} = 0, \tag{9}$$
$$M_{xx}^{(1)} + M_{xx}^{(2)} = 0, \tag{10}$$
$$(N_{xx}^{(1)} - N_{xx}^{(2)})\sin\alpha - (V_x^{(1)} + V_x^{(2)})\cos\alpha = 0, \tag{11}$$
$$(N_{xx}^{(1)} + N_{xx}^{(2)})\cos\alpha + (V_x^{(1)} - V_x^{(2)})\sin\alpha = 0, \tag{12}$$
$$N_{xy}^{(1)} + N_{xy}^{(2)} = 0, \tag{13}$$

where the superscripts 1 and 2 refer to the two sheets shown in Fig. 1(b), and V_x denotes the supplemented shearing force that includes the derivative of the twisting moment. The first four boundary conditions pertain to the requirement of continuous deformations, while the last four enforce Newton's third law. In considering the three specific cases of loading indicated in Fig. 1(a), the last three boundary conditions must be appropriately supplemented with delta functions to incorporate the given concentrated forces in the formulation. It may also be noted that the first three boundary conditions must be satisfied only within arbitrary rigid-body displacements. Therefore (6) and (7) can be differentiated twice and (8) once with respect to y at no loss of conditions that determine the solution.

The boundary conditions at infinity can neither be specified by a physical argument. nor predicted in advance on mathematical grounds without attempting to solve the problem. However, this question can be left open for the time being, as the proper, and in fact the only admissible boundary conditions evolve in the course of solution.

The problem proposed yields readily to a routine application of the Fourier exponential transform [8]. Since numerous books and papers give sufficient details for mathematically analogous problems, this part of the analysis can safely be omitted. It may only be mentioned that the program is always the same: applying the Fourier exponential transform with respect to y reduces the partial differential equations (3) and (5) to simple ordinary differential equations that can be integrated. This yields the transforms of φ, ψ and w in terms of the still unknown integration constants. Next, the transforms are applied to the boundary conditions. Substitution of the transforms of φ, ψ and w into the transformed boundary conditions leads to simultaneous algebraic equations for the integration constants. Once the integration constants have been determined, the quantities φ, ψ and w can be obtained from the inversion formula for the Fourier transform. Since it can be verified by a direct substitution that the solutions satisfy the field equations and the boundary conditions, only the final results will be given.

The present problem includes the completely folded plate ($\alpha = 0$) and the flat plate ($\alpha = \pi/2$) as limiting cases. It can be expected at the outset, therefore, that the Fourier exponential transform will yield divergent integrals for some of the field quantities, such as the extensional displacements and the transverse deflection. However, if the divergent integrals are replaced by certain elementary functions, it can be verified that the resulting fields satisfy the differential equations (3) and (5), and the boundary conditions (6)–(13).

AUXILIARY FUNCTIONS

Although the field quantities, for which the Fourier transform yields convergent inverses, can be expressed through exponential integrals of complex arguments, it is convenient to introduce the following four higher transcendental functions:

$$\begin{aligned} C_0(x, y; a) &= \int_0^\infty \frac{e^{-xt} \cos yt}{t^2 + a^2} \, dt, \\ S_0(x, y; a) &= \int_0^\infty \frac{e^{-xt} \sin yt}{t^2 + a^2} \, dt, \\ C_1(x, y; a) &= \int_0^\infty \frac{t e^{-xt} \cos yt}{t^2 + a^2} \, dt, \\ S_1(x, y; a) &= \int_0^\infty \frac{t e^{-xt} \sin yt}{t^2 + a^2} \, dt, \end{aligned} \tag{14}$$

with $x > 0$, $-\infty < y < \infty$ and $a^2 > 0$. The derivatives of these auxiliary functions are

$$\begin{aligned}
\frac{\partial C_0}{\partial x} &= -C_1, & \frac{\partial C_0}{\partial y} &= -S_1,\\
\frac{\partial S_0}{\partial x} &= -S_1, & \frac{\partial S_0}{\partial y} &= C_1,\\
\frac{\partial C_1}{\partial x} &= -\frac{x}{r^2}+a^2C_0, & \frac{\partial C_1}{\partial y} &= -\frac{y}{r^2}+a^2S_0,\\
\frac{\partial S_1}{\partial x} &= -\frac{y}{r^2}+a^2S_0, & \frac{\partial S_1}{\partial y} &= \frac{x}{r^2}-a^2C_0,
\end{aligned} \tag{15}$$

$$\begin{aligned}
\frac{\partial^2 C_0}{\partial x^2} &= \frac{x}{r^2}-a^2C_0, & \frac{\partial^2 C_0}{\partial x\partial y} &= \frac{y}{r^2}-a^2S_0,\\
\frac{\partial^2 S_0}{\partial x^2} &= \frac{y}{r^2}-a^2S_0, & \frac{\partial^2 S_0}{\partial x\partial y} &= -\frac{x}{r^2}+a^2C_0,\\
\frac{\partial^2 C_1}{\partial x^2} &= -\frac{1}{r^2}+\frac{2x^2}{r^4}-a^2C_1, & \frac{\partial^2 C_1}{\partial x\partial y} &= \frac{2xy}{r^4}-a^2S_1,\\
\frac{\partial^2 S_1}{\partial x^2} &= \frac{2xy}{r^4}-a^2S_1, & \frac{\partial^2 S_1}{\partial x\partial y} &= \frac{1}{r^2}-\frac{2x^2}{r^4}+a^2C_1,\\
\frac{\partial^2 C_0}{\partial y^2} &= -\frac{x}{r^2}+a^2C_0, &&\\
\frac{\partial^2 S_0}{\partial y^2} &= -\frac{y}{r^2}+a^2S_0, &&\\
\frac{\partial^2 C_1}{\partial y^2} &= \frac{1}{r^2}-\frac{2x^2}{r^4}+a^2C_1, &&\\
\frac{\partial^2 S_1}{\partial y^2} &= -\frac{2xy}{r^4}+a^2S_1, &&
\end{aligned} \tag{16}$$

where $r^2 = x^2+y^2$. It is seen from these expressions that the auxiliary functions are harmonic.

Combining the auxiliary functions pairwise into functions of the complex argument $z = x+iy$, they can be related to cosine and sine integrals.† These may in turn be expressed through the exponential integral of a complex argument for which numerical tables are available [10]. Thus,

$$\begin{aligned}
2aC_0 &= -\mathrm{Im}\{e^{iaz}E_1(iaz)-e^{-iaz}E_1(-iaz)\},\\
2aS_0 &= -\mathrm{Re}\{e^{iaz}E_1(iaz)-e^{-iaz}E_1(-iaz)\},\\
2C_1 &= \mathrm{Re}\{e^{iaz}E_1(iaz)+e^{-iaz}E_1(-iaz)\},\\
2S_1 &= -\mathrm{Im}\{e^{iaz}E_1(iaz)+e^{-iaz}E_1(-iaz)\},
\end{aligned} \tag{17}$$

† See Ref. [9, No. 4.2.14].

where [10]

$$E_1(z) = \int_z^\infty t^{-1}e^{-t}\,dt, \qquad |\arg z| < \pi, \tag{18}$$

with the restriction that the path of integration in (18) neither passes through the origin. nor crosses the negative real axis.

The asymptotic forms of the four auxiliary functions for $r \to 0$ and $r \to \infty$ could be obtained from the known properties of the exponential integral, but they are equally easy to deduce directly. For instance, with $x = r\cos\theta = rp$ and $y = r\sin\theta = rq$,

$$C_1(x, y) = \int_0^\infty \frac{te^{-pt}\cos qt}{t^2+a^2r^2}\,dt. \tag{19}$$

Rewriting (19) as

$$C_1(x, y) = \int_0^1 \frac{t}{t^2+a^2r^2}\,dt - \int_0^1 \frac{t(1-e^{-pt}\cos qt)}{t^2+a^2r^2}\,dt + \int_1^\infty \frac{te^{-pt}\cos qt}{t^2+a^2r^2}\,dt, \tag{20}$$

it is seen that the last two integrals remain bounded as $r \to 0$, while the first integral can be evaluated to yield $-\log(ar)$ in the limit. Thus, for $r \to 0$,

$$\begin{aligned} C_0(x, y) &= O(1), \qquad C_1(x, y) = -\log r + O(1), \\ S_0(x, y) &= o(1), \qquad S_1 = \theta = \arctan(y/x). \end{aligned} \tag{21}$$

It may also be noted that, for $r \to 0$, the dominant parts in the derivatives of the four auxiliary functions are the same as the derivatives of certain elementary harmonic functions. The correspondences are

$$\begin{aligned} &C_0(x, y): x\log r - y\theta, \\ &S_0(x, y): -(x\theta + y\log r), \\ &C_1(x, y): -\log r, \\ &S_1(x, y): \theta. \end{aligned} \tag{22}$$

Using the identity

$$\frac{1}{t^2+a^2r^2} = \frac{1}{a^2r^2} - \frac{t^2}{a^2r^2(t^2+a^2r^2)}, \tag{23}$$

which is obtained by dividing $a^2r^2+t^2$ into 1, we have from (19)

$$C_1(x, y) = \frac{1}{a^2r^2}\int_0^\infty te^{-pt}\cos qt\,dt - \frac{1}{a^2r^2}\int_0^\infty \frac{t^3e^{-pt}\cos qt}{t^2+a^2r^2}\,dt. \tag{24}$$

The first integral in (24) can be evaluated to yield $(-1+2p^2)$. The second integral vanishes as $r \to \infty$ and, therefore, the last term in (24) is of $o(r^{-2})$. If, however, the division in (23) were carried one step further, the last term in (24) would be seen to be of $O(r^{-4})$. Additional terms in the expansion can be obtained by continuing the process of division indicated in (23). This approach leads to the following results for $r \to \infty$:

$$\begin{aligned}
a^2C_0(x, y) &= \frac{x}{r^2}+\frac{2}{a^2}\left(\frac{3x}{r^4}-\frac{4x^3}{r^6}\right)+O(r^{-5}),\\
a^2S_0(x, y) &= \frac{y}{r^2}+\frac{2}{a^2}\left(\frac{y}{r^4}-\frac{4x^2y}{r^6}\right)+O(r^{-5}),\\
a^2C_1(x, y) &= -\frac{1}{r^2}+\frac{2x^2}{r^4}-\frac{6}{a^2}\left(\frac{1}{r^4}-\frac{8x^2}{r^6}+\frac{8x^4}{r^8}\right)+O(r^{-6}),\\
a^2S_1(x, y) &= \frac{2xy}{r^4}+\frac{24}{a^2}\left(\frac{xy}{r^6}-\frac{2x^3y}{r^8}\right)+O(r^{-6}).
\end{aligned} \tag{25}$$

It can also be shown that not only the dominant parts of the auxiliary functions themselves, but also those of their derivatives for $r \to \infty$ are identical with the derivatives of elementary harmonic functions according to the scheme:

$$\begin{aligned}
&a^2C_0(x, y): \cos\theta/r,\\
&a^2S_0(x, y): \sin\theta/r,\\
&a^2C_1(x, y): \cos 2\theta/r^2,\\
&a^2S_1(x, y): \sin 2\theta/r^2.
\end{aligned} \tag{26}$$

CASE I—SYMMETRIC LOAD PERPENDICULAR TO RIDGE

Suppose the shell is loaded by the force marked P_1 in Fig. 1(a) which lies in the center plane and is perpendicular to the ridge. Because of the symmetric nature of the deformations about the center plane we have

$$\begin{aligned}
u_x^{(1)}(x_1, y_1) &= u_x^{(2)}(x_2, y_2) = u_x(x, y),\\
u_y^{(1)}(x_1, y_1) &= u_y^{(2)}(x_2, y_2) = u_y(x, y),\\
w^{(1)}(x_1, y_1) &= -w^{(2)}(x_2, y_2) = w(x, y).
\end{aligned} \tag{27}$$

Four of the boundary conditions (6)–(13) are then satisfied identically, while the remaining four become

$$\begin{aligned}
u_x \sin\alpha - w\cos\alpha &= 0,\\
\frac{\partial w}{\partial x} &= 0,\\
N_{xx}\cos\alpha + V_x \sin\alpha &= -\tfrac{1}{2}P_1\delta(y),\\
N_{xy} &= 0.
\end{aligned} \tag{28}$$

A formal application of the Fourier exponential transform to the problem leads to expressions that contain the divergent integrals $\int_0^\infty t^{-1}\,e^{-xt}\cos yt\,dt$ and $\int_0^\infty t^{-2}\,e^{-xt}$

$\cos yt\,dt$ which cannot be tolerated in the result. How to proceed is suggested by first considering the function

$$c_{-1}(x, y; \varepsilon) = \int_{\varepsilon}^{\infty} t^{-1} e^{-xt} \cos yt\, dt, \tag{29}$$

with $x > 0$, $-\infty < y < \infty$, $\varepsilon > 0$. Differentiating this function we obtain integrals that can be evaluated. Thus,

$$\begin{aligned} \frac{\partial c_{-1}}{\partial x} &= -\frac{e^{-\varepsilon x}}{r^2}(x \cos \varepsilon y - y \sin \varepsilon y), \\ \frac{\partial c_{-1}}{\partial y} &= -\frac{e^{-\varepsilon x}}{r^2}(y \cos \varepsilon y + x \sin \varepsilon y), \end{aligned} \tag{30}$$

where $r^2 = x^2 + y^2$. Furthermore,

$$\begin{aligned} \lim_{\varepsilon \to 0} \frac{\partial c_{-1}}{\partial x} &= -\frac{x}{r^2} = -\frac{\partial}{\partial x} \log r, \\ \lim_{\varepsilon \to 0} \frac{\partial c_{-1}}{\partial y} &= -\frac{y}{r^2} = -\frac{\partial}{\partial y} \log r. \end{aligned} \tag{31}$$

This suggests that the first divergent integral be replaced by $-\log r$. Next considering the second derivatives of the function

$$c_{-2}(x, y; \varepsilon) = \int_{\varepsilon}^{\infty} t^{-2} e^{-xt} \cos yt\, dt, \tag{32}$$

where $x > 0$, $-\infty < y < \infty$ and $\varepsilon > 0$, it is equally suggestive to replace the second divergent integral by the function $x(\log r - 1) - y\theta$, where $\theta = \arctan(y/x)$. When this is done, the following results emerge:

$$\varphi(x, y) = -\frac{P_1}{2\pi h \cos \alpha}[\log r + C_1(x, y; a)], \tag{33}$$

$$\psi(x, y) = -\frac{P_1(1-\nu)}{2\pi(1+\nu)\cos \alpha}[x(\log r - 1) - y\theta - C_0(x, y; a)], \tag{34}$$

$$w(x, y) = -\frac{P_1 \sin \alpha}{2\pi(1+\nu)Gh \cos^2\alpha}[\log r + C_1(x, y; a) - a^2 x C_0(x, y; a)], \tag{35}$$

where

$$a^2 = 3(1-\nu^2)\cot^2\alpha. \tag{36}$$

Since the Fourier exponential transform yielded divergent integrals which were replaced in a somewhat arbitrary fashion by certain elementary functions, there is no guarantee that (33)–(35) is the solution of the given problem. It is readily verified by direct substitution, however, that (33)–(35) satisfy the field equations (3) and (5), as well as the boundary conditions (28) and, thus, constitute the desired solution.

The extensional displacements derived from (33) and (34) are

$$2Gu_x = \frac{P_1}{2\pi(1+\nu)h\cos\alpha}[-2(\log r + C_1) + (1+\nu)a^2 x C_0],$$
$$2Gu_y = \frac{P_1}{2\pi(1+\nu)h\cos\alpha}[(1-\nu)(S_1 - \theta) + (1+\nu)a^2 x S_0], \tag{37}$$

where $-\pi/2 \leq \theta \leq \pi/2$. The membrane forces follow from (2) as

$$N_{xx} = -\frac{P_1 a^2}{2\pi h\cos\alpha}(C_0 + xC_1),$$
$$N_{xy} = -\frac{P_1 a^2}{2\pi h\cos\alpha} x S_1, \tag{38}$$
$$N_{yy} = -\frac{P_1 a^2}{2\pi h\cos\alpha}(C_0 - xC_1).$$

The membrane forces can also be obtained from the Airy stress function

$$\chi = -\frac{P_1 h}{2\pi\cos\alpha}(y\theta + C_0 + xC_1). \tag{39}$$

Finally, the transverse deflection given by (35) yields the following expressions for the stress resultants of the bending field:

$$M_{xx} = \frac{P_1}{4\pi\sin\alpha}\left[(1+\nu)C_1 - (1-\nu)\left(\frac{x^2}{r^2} - a^2 x C_0\right)\right],$$
$$M_{xy} = \frac{P_1(1-\nu)}{4\pi\sin\alpha}\left(\frac{xy}{r^2} - a^2 x S_0\right), \tag{40}$$
$$M_{yy} = \frac{P_1}{4\pi\sin\alpha}\left[(1+\nu)C_1 + (1-\nu)\left(\frac{x^2}{r^2} - a^2 x C_0\right)\right],$$

$$Q_x = \frac{P_1}{2\pi h\sin\alpha}\left(-\frac{x}{r^2} + a^2 C_0\right),$$
$$Q_y = \frac{P_1}{2\pi h\sin\alpha}\left(-\frac{y}{r^2} + a^2 S_0\right). \tag{41}$$

The limiting cases of $\alpha = 0$ and $\alpha = \pi/2$ are special and need to be considered separately.

Limit of $\alpha = 0$

Noting that

$$\lim_{\alpha\to 0} a^2 C_0 = \frac{x}{r^2},$$
$$\lim_{\alpha\to 0} a^2 S_0 = \frac{y}{r^2},$$
$$\lim_{\alpha\to 0} a^2 C_1 = -\frac{1}{r^2} + \frac{2x^2}{r^4}, \tag{42}$$
$$\lim_{\alpha\to 0} a^2 S_1 = \frac{2xy}{r^4},$$

and that C_0, S_0, C_1, S_1, $\sin\alpha a^2 C_0$, $(\sin\alpha)^{-1}C_1$, $(\sin\alpha)^{-1}[(x/r^2)-a^2C_0]$ and $(\sin\alpha)^{-1}[(y/r^2)-a^2S_0]$ all vanish as $\alpha\to 0$, the field quantities can be evaluated without difficulty. The results that emerge are precisely what one would expect on physical grounds: the bending field vanishes, while the membrane field is the same as given by the Flamant solution [11] for an elastic half-plane subjected to the normal force $P_1/2$.

Limit of $\alpha = \pi/2$

The limiting case when the shell degenerates into a flat plate is not nearly as well behaved as the previous one, because most of the field quantities diverge. An exception is provided by the transverse shearing forces given by (41), which approach the proper values for these quantities in the vicinity of a concentrated force acting perpendicularly to a flat plate. More satisfactory results for this limiting case could be obtained in the framework of distributions or generalized functions, but the returns would be marginal for our purposes.

In view of the foregoing, further discussion is restricted to $0 < \alpha < \pi/2$.

THE NEAR AND FAR FIELDS FOR CASE I

The near ($r\to 0$) and far ($r\to\infty$) fields of the stress resultants are obtained from the general expressions (38), (40) and (41) by substituting the asymptotic forms (21) and (25) for the auxiliary functions.

Using the superscript 0 for the quantities of the near field, we have

$$\begin{aligned} N_{xx}^0 &= N_{yy}^0 = O(1), \\ N_{xy}^0 &= O(r), \end{aligned} \tag{43}$$

$$\begin{aligned} M_{xx}^0 = M_{yy}^0 &= -\frac{P_1(1+\nu)}{4\pi\sin\alpha}\log r + O(1), \\ M_{xy}^0 &= \frac{P_1(1-\nu)}{4\pi\sin\alpha}\cdot\frac{xy}{r^2} + o(1), \end{aligned} \tag{44}$$

$$\begin{aligned} Q_x^0 &= -\frac{P_1}{2\pi h\sin\alpha}\cdot\frac{x}{r^2} + O(1), \\ Q_y^0 &= -\frac{P_1}{2\pi h\sin\alpha}\cdot\frac{y}{r^2} + O(1). \end{aligned} \tag{45}$$

The dominant parts of the bending stress resultants are the same as the stress resultants derived from the elementary biharmonic function

$$w^0 = \frac{P_1 h^2}{8\pi D\sin\alpha} r^2 \log r. \tag{46}$$

It may be noted that the bending quantities in the near field agree with the known results also in the case of $\alpha = \pi/2$.

The membrane forces at the origin can be evaluated, since $C_0(0, 0; a)$ is an elementary integral. Thus,

$$N_{xx}(0,0) = N_{yy}(0,0) = -\frac{P_1 a}{4h\cos\alpha}, \qquad N_{xy}(0,0) = 0. \tag{47}$$

Labelling the quantities of the far field with the superscript ∞, the membrane forces are

$$\begin{aligned} N_{xx}^{\infty} &= -\frac{P_1}{\pi h\cos\alpha}\cdot\frac{x^3}{r^4}+O(r^{-3}),\\ N_{xy}^{\infty} &= -\frac{P_1}{\pi h\cos\alpha}\cdot\frac{x^2 y}{r^4}+O(r^{-3}),\\ N_{yy}^{\infty} &= -\frac{P_1}{\pi h\cos\alpha}\left(\frac{x}{r^2}-\frac{x^3}{r^4}\right)+O(r^{-3}). \end{aligned} \tag{48}$$

The dominant parts of the membrane forces in (48) can be derived from the Airy stress function

$$\chi^{\infty} = -\frac{P_1 h}{2\pi\cos\alpha}y\theta. \tag{49}$$

For the far field of the bending stress resultants we obtain

$$\begin{aligned} M_{xx}^{\infty} &= \frac{P_1}{4\pi a^2\sin\alpha}\left[-(1+\nu)\frac{1}{r^2}+4(2-\nu)\frac{x^2}{r^4}-8(1-\nu)\frac{x^4}{r^6}\right]+O(r^{-4}),\\ M_{xy}^{\infty} &= -\frac{P_1(1-\nu)}{2\pi a^2\sin\alpha}\left(\frac{xy}{r^4}-\frac{4x^3y}{r^6}\right)+O(r^{-4}),\\ M_{yy}^{\infty} &= -\frac{P_1}{4\pi a^2\sin\alpha}\left[(1+\nu)\frac{1}{r^2}+4(1-2\nu)\frac{x^2}{r^4}-8(1-\nu)\frac{x^4}{r^6}\right]+O(r^{-4}), \end{aligned} \tag{50}$$

$$\begin{aligned} Q_x^{\infty} &= \frac{P_1}{\pi a^2 h\sin\alpha}\left(\frac{3x}{r^4}-\frac{4x^3}{r^6}\right)+O(r^{-5}),\\ Q_y^{\infty} &= \frac{P_1}{\pi a^2 h\sin\alpha}\left(\frac{y}{r^4}-\frac{4x^2y}{r^6}\right)+O(r^{-5}). \end{aligned} \tag{51}$$

The dominant parts of the bending stress resultants can be derived from the elementary deflection function

$$w^{\infty} = -\frac{P_1 h^2}{4\pi a^2 D\sin\alpha}(\log r-\tfrac{1}{2}\cos 2\theta). \tag{52}$$

DISCUSSION OF RESULTS FOR CASE I

Several interesting features in the behavior of the prismatic shell emerge immediately from the asymptotic expansions of the stress resultants. Perhaps the most unexpected result which seems to contradict intuition is that the membrane forces remain bounded at the point of application of the force P_1 for all $\alpha > 0$. A physical explanation of this aspect of the results can be devised, but the discussion of this point is better taken up after studying the antisymmetric force P_2 applied perpendicularly to the ridge. Equations (43)–(45) show that, while the membrane forces are of $O(1)$, the moments are of $O(\log r)$ and the transverse shearing forces of $O(r^{-1})$ near the origin. The orders of the various near-field quantities imply physically that the concentrated force P_1 is transmitted into the shell essentially through bending action. Although the present analysis is not expected to give accurate results near the ridge, especially for small α, it is reasonable to anticipate that this general conclusion remains qualitatively correct in a real prismatic shell. The advice to the designer is, therefore, that in the vicinity of columns a prismatic shell be principally reinforced for the transmission of moments and transverse shearing forces rather than membrane forces.

In a sense, exactly the opposite physical response of the prismatic shell emerges from the expressions for the far field. Equations (48), (50) and (51) reveal that the membrane forces, being of $O(r^{-1})$, decay much slower than the moments and transverse shearing forces, which are of $O(r^{-2})$ and $O(r^{-3})$, respectively. We can conclude, therefore, that the applied force P_1 is diffused far away essentially by membrane action. It should be realized, of course, that in a real case the far field is strongly affected, if not overwhelmed, by the effect of supports. However, the trends predicted by the analysis of the far field are again suggestive for design, especially the types of supports that would minimize bending.

The nature of the fields at intermediate positions can only be judged by evaluating the auxiliary functions and representing the results graphically. In view of the large number of field quantities involved in the problem, no parametric study for various values of α and ν is attempted in the present context, and a few results are presented merely for the purpose of conveying some idea about the more important stress resultants. Figure 2 shows the

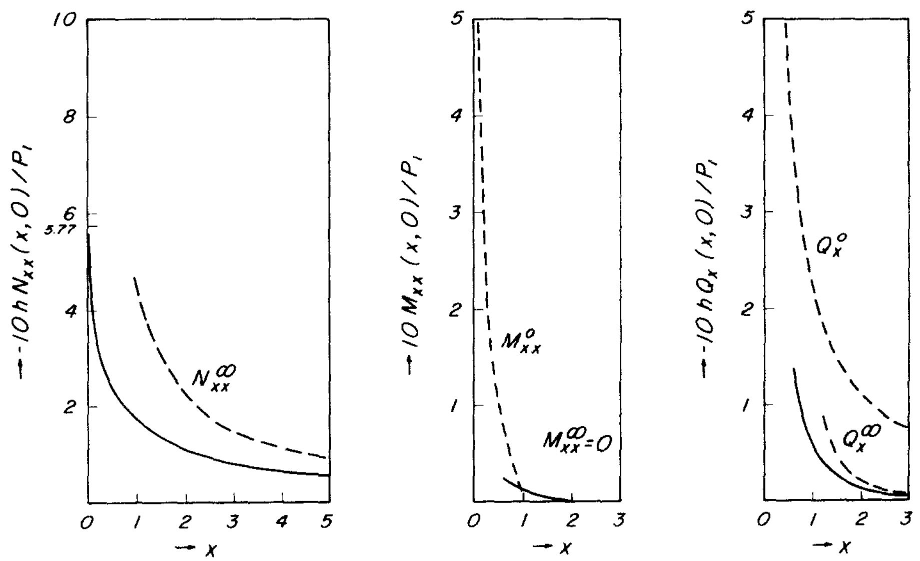

FIG. 2. Stress resultants N_{xx}, M_{xx} and Q_x along the x-axis for $\alpha = 45°$ and $\nu = 0{\cdot}3$.

variation of the membrane force N_{xx}, bending moment M_{xx} and transverse shearing force Q_x along the axis of symmetry, or $y = 0$, for the case of $\alpha = 45°$ and $\nu = 0{\cdot}3$. For the same values of α and ν, the membrane force and bending moment transmitted by a section parallel to the ridge are depicted in Fig. 3. Where appropriate, the near and far field approximations are also indicated in both of these figures. Perhaps the most noteworthy feature of the results is that the near field expansions do not approximate the stress resultants well at distances from the ridge where the solution can be expected to be numerically accurate. In contrast, the far field expansions do better than might be expected on physical grounds. Thus at distances from the origin equal to ten to twenty times the thickness, the far field expansions provide excellent approximations for the stress resultants.

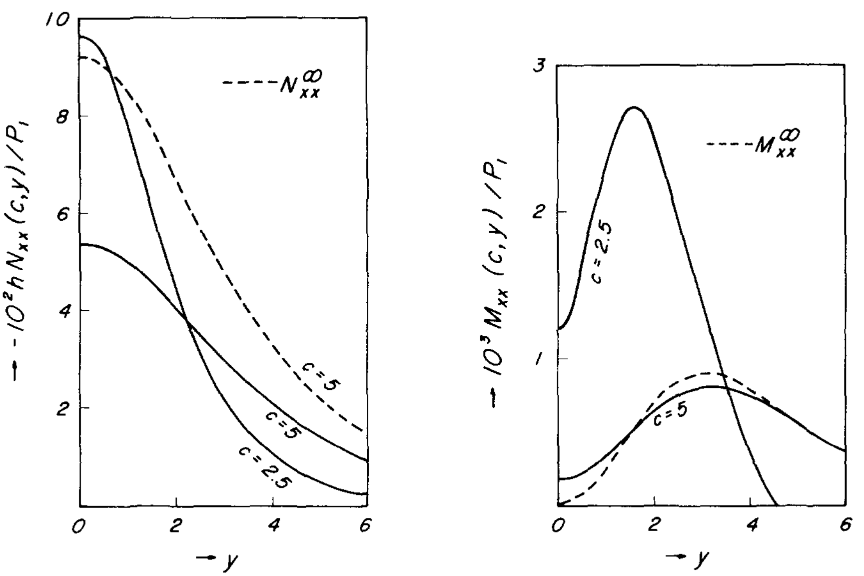

FIG. 3. Stress resultants N_{xx} and M_{xx} along a line parallel to the ridge for $\alpha = 45°$ and $\nu = 0{\cdot}3$.

Acknowledgements—The results reported here were obtained in the course of research supported by the National Science Foundation under Grant GK-2787. One of the authors (J.D.) would like to thank Prof. M. Stippes at the University of Illinois for a clarifying discussion on the higher transcendental function encountered in the analysis.

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(*Received* 5 *March* 1971)

Абстракт—Работа рассматривает вопрос полей безмоментного и моментного состояний, вызванных в призматических оболочках нагруженных концентритическими силами, приложенными к связям и ребрам между двумя пластинками. В виду того, что основные задачи мало связаны с условиями операния или конечными границами, рассматривается неограниченная оболочка, состоящаяся из двух полубесконечных пластинок. Такая геометрия приводит к прямому анализу преобразованиях фурье.

Результаты можна выразить в форме зкспотенциальных интегралов комплексного аргумента, для которых существуют таблицы. Когда сосредоточная сила действует в плоскости симметрии и перпедикулярна к ребру,тогда появляется неожидаемый результат, состоящий в том, что силы в безмоментном состоянии ограниченные в точке приложения нагрузки. В противоположность, моменты и поперечные силы сингулярны в точке нагрузки. Асимотические выражения, для полей на далеком растоянии, указывают, что мембранные усилия уменьщаются более медленно, чем моменты и поперечные силы. Зти состояния для близких и далеких полей указывают, что симметрически приложенные перпендикулярные силы переходит в призметическую оболочку, главным обазом нутем изгиба и исезают далеко в безмоментном состоянии.